Weak measurements of non local variables

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Abstract

Methods for measuring the weak value of non local variables are investigated. We analyze local (indirect) measurement methods for obtaining the weak values. We also describe some new (direct) methods (*Non local weak measurements*) for measuring the weak values of some non local variables.

Non local variables are variables of a composite system with parts in two or more remote locations. An example of a non local variable is the sum of the spin components of a spin singlet state with one spin 1/2 particle on earth and the other on the moon. Non local variables cannot always be

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measured in a direct way because such measurements can sometimes contradict relativistic causality. Those non local variables that can be measured require the use of a specific, entangled measuring device for the (non local) measurement .

Weak measurements are best described as standard measurements with a weakened coupling. Unlike the standard (strong) measurement whose result is one of the eigenvalues, the result of weak measurements is the *weak value* , a complex number. Weak measurements have an inherently large uncertainty, and must therefore be made on a large ensemble of identical systems. The weak values under investigation here are those of a non local sum and a non local product of two observables.

For the sum of two observables we describe a method for measuring the weak value in a direct way. We also describe an indirect method for obtaining the weak value using the relation between the sum of the local weak values and the weak value of the non local sum. This indirect method requires two local measuring devices, one at each location. The indirect (local) method is compared to the direct (non local) method and is shown to be inefficient.

For the product of two observables, there is no direct method for measuring the non local weak values. In some special cases the methods we found for the measurement of a sum can be used to measure the product. Using these methods to measure the product requires some prior knowledge of the system to be measured.

We give a critical analysis of a recent method for measuring the product (called "*joint weak values*"). This method for obtaining the weak value of the product using local weak measurements is shown to be even more ineffi-

cient than the local method described for the sum. We compare this method with a (non physical) direct method and show that it requires much larger resources.

Contents

1 Introduction

Quantum measurement theory gives us the tools required for investigating the limits of measurements under the quantum regime. The limits on "What can be measured?" and "with what precision?" , have been under investigation from the first days of quantum theory and are still being discussed today. [1, 2, 3]. One of the fundamental questions relating to our understanding of the quantum world , the implementation of quantum computers and to quantum information theory is that of the instantaneous measurement[1] of non local variables.

Non local variables are variables of a composite system with parts in two or more remote locations. An example is the sum of the spin components of a singlet state $\frac{|\uparrow\downarrow\rangle_{AB} - |\downarrow\uparrow\rangle_{AB}}{\sqrt{2}}$ where particle A (belonging to Alice) is on the moon while particle B (belonging to Bob) is here on earth. Such variables were thought, at first, to be unmeasurable (in an instantaneous non-demolition measurement)[1] since they seem to contradict relativistic causality. However it has been shown by Aharonov Albert and Vaidman that some non local variables can be measured instantaneously. [2, 4, 5]. As will be shown later, such measurements require a very specific preparation of a non local measuring device (i.e an entangled system). We will look at two types of non local variables . The sum of two observables (eg.$\sigma_z^A + \sigma_z^B$) and the product of two observables (eg.$\sigma_z^A \sigma_z^B$).

There are methods for making non demolition measurements of the sum, using an entangled measuring device prepared in a very specific way. These methods can be used for making weak measurements if the coupling is weakened. Another non local variable which can be measured in an instantaneous non demolition

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[1]Measurement here and throughout refers to an instantaneous measurement

measurement is the modular sum (eg.$(\sigma_z^A + \sigma_z^B) \ mod \ 4$). This measurement requires the use of an entangled measuring device with a very strict periodic nature which cannot be used for pointing at weak values.

The measurement of a product in an instantaneous non demolition measurement can sometimes allow us to send superluminal signals and thus contradict relativistic causality. But the product can be measured in a demolition measurement, i.e one where the final state of the system, after measurement, is not the eigenstate corresponding to the result [2].

All measurements require some coupling between a measuring device and the system to be measured. Van Neumann measurements[6] are usually used to describe the measurement of a variable A in the following way:

The measuring device is described by a pointer variable Q and it's conjugate momentum P. It is coupled to the system via a coupling Hamiltonian.

$$H_M = g(t)PA \tag{1}$$

Where $g(t)$ is large for a very short time around the measurement.

When this coupling is very weak we have a new type of measurement called a *weak measurement*[7, 8]. Weak measurements were shown to be of special interest when discussing pre and post selected systems, systems with a definite past and a definite future as described by a "Two state vector" (TSV) [9, 8]. When measuring these pre and post selected systems using weak measurements, the result is something new called the *Weak Value* defined as:

$$(A_w) \equiv \frac{\langle \Phi | \, A \, | \Psi \rangle}{\langle \Phi | \Psi \rangle} \tag{2}$$

This is the weak value of the observable A belonging to a system pre selected as $|\Psi\rangle$ and post selected as $|\Phi\rangle$.

One of the unique properties of weak values is that they can be far outside the range of allowed eigenvalues. The weak measurement of the spin of an electron can give the result 100. This strange and surprising effect occurs when the pre and post selection are almost orthogonal. The pointer variable of the measuring device is shifted by this weak value as long as the coupling is weak. We can think of the measuring device as being effectively coupled to the weak value.

We can now begin our discussion of non local weak measurements and the measurement of non local weak values. Values of non local observables can essentially be measured in two ways:

1) A direct measurement of the non local variable (a non local measurement)

2) An indirect measurement of the non local variable using local measurements only. (often more than one measurement, requiring an ensemble of identically prepared systems).

The second method is often very disturbing, since the measurement results in a projection to local product states. Seemingly weak measurements are an ideal way to overcome this problem since they are inherently non disturbing. On the other hand it seems that weak measurements may not be effected by non local correlations for the same reason. Nevertheless a recent method for measuring the weak value of a non local product $(AB)_w$ (of two observables A and B) makes use of the correlations between the local measuring devices [10]. This method called "*Joint weak values*" uses second order effects to show that it is possible to calculate the weak value by joining the results of two local weak measurements. Actually rather then look at the pointer of each measuring device (Q_A ; Q_B). we

must look at correlations between the two measuring devices $(Q_A Q_B)$. If both measuring devices are prepared in a certain way we get the final result.

$$Re(AB)_w = 2\langle Q_A Q_B \rangle - Re(A_w^* B_w) \tag{3}$$

As can be seen from the formula above, this is not a direct measurement of the variable $(AB)_w$. In what follows we show that there is still a major difference between such local weak measurements and a theoretical (non-physical) non local weak measurement. We can of course still ask ourselves if non local weak values can be measured using a true non local weak measurement method. The problem with constructing such non local weak measurements is that they have to follow the restrictions of both non local measurements and weak measurements. In some cases as in that of the modular sum these restrictions seem to be in direct conflict with each other. So can we measure non local weak values? can we do so using local weak measurements ? And is there a direct method for obtaining the non local weak value? These are the questions we want to answer.

In the next three sections we introduce the basic concepts required to answer the above questions: non local measurements and non local variables, the two state vector formalism, and weak measurements. Throughout we assume that a measuring device (Q, P) can be coupled to a physical system via a local operator A using the coupling Hamiltonian $H_M = g(t)PA$. Under this assumption we can measure all local variables (weak or strong) directly. Quantum mechanics teaches us that all measurements affect the system, we will discuss these effects for both weak and strong measurements. When speaking of weak measurements we define a new variable called the weak uncertainty (the uncertainty in the weak value). We

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also describe an interesting property of weak measurements. Weak measurements can be made on an ensemble of random states, as long as the weak value to be measured is the same for all systems in the ensemble. This is a consequence of the effective coupling between the measuring device and the weak value.

In section 5 we discuss non local measurements under the two state vector formalism and define a general classification for an efficient and inefficient measurement using the uncertainty. We will then go on to deal with non local weak measurements and measurements of Kent Naganonon local weak values.

In section 6 we discuss the measurement of the simplest non local weak value, that of a sum of two observables. For this simple case, there exists a simple relation between the local and non local weak values. $A_w + B_w = (A + B)_w$. We show how this can be used to make an indirect measurement of the weak value of the sum. We also describe non local weak measurement methods for pointing at the weak value . These measurement methods are then compared for the general case and for some specific examples. It is shown that the local method is an inefficient measurement method.

In section7 we discuss the method of joint weak measurements introduced by Resch and Steinberg [10, 11] and later refined by Resch and Lundeen [12]. We analyze this method and compare it to a non-physical non local method. It is shown that such a measurement is again inefficient in all cases. This method requires such large resources that it seems to be very unpractical in most interesting physical situations. We also look at sequential weak measurements [13], and sequential weak values. The sequential weak values are the weak values of the product of two operators at different times $(B_{t_2} A_{t_1})_w = \frac{\langle \Phi | B_{t_2} V A_{t_1} | \Psi \rangle}{\langle \Phi | \Psi \rangle}$ where V is the unitary evolution between the times t_1 and t_2. These sequential weak values are of

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some interest when looking at systems where there is no meaning for a sequential "strong value". An example of this is a double interferometer experiment where one measurement at time t_1 effects the outcome of a subsequent measurement at time t_2. Since there is no direct way of measuring the sequential weak values, their status as "elements of reality" is questioned. Finally we show that the method of joint weak measurements cannot be used to measure the weak value of a random ensemble of system with the same weak value.

In the last section (8) we discuss the meaning of weak values. We show that in the standard weak measurement process the measuring device is effectively coupled to the weak value. We discuss other measurement schemes for obtaining the weak value and show that for such measurements the weak value has a lesser status since it is just the result of a bra-ket calculation.

2   Non local measurements and non local variables.

Non local variables are unique to the quantum world. The fact that such variables can be measured directly even in theory is not trivial since such measurements can often contradict relativistic causality (by allowing us to send a superluminal signal). Still some non local variables can be measured directly as will be shown in this section. But before we begin our description of non local measurements, let us first review some of the principles of "ideal" local measurement.

Ideal (non demolition) measurement as described by Von Neumann[6] (also called a Von Neumann measurement) consists of a coupling interaction between a measuring device with a pointer variable Q and its conjugate momentum P and a system $|\Psi\rangle$ via a coupling, (Von Neumann) Hamiltonian.

$$H_M = g(t)PA \tag{4}$$

where A is the observable we want measured and $g(t)$ is non zero only for a short time about the measurement. For such a measurement to be ideal, the measuring device and the system should end up (after the interaction) in an entangled state. So that eigenstates of the same eigenvalue will correspond to orthogonal states of the measuring device and the amplitudes corresponding to each result remain the same. For a state described by

$$|\Psi\rangle_i = \sum \alpha_i |A = a_i\rangle \tag{5}$$

and a measuring device $|0\rangle_{MD}$ we should end up with

$$|\Psi MD\rangle_f = \sum \alpha_i |A = a_i\rangle |a_i\rangle_{MD} \tag{6}$$

where all $|a_i\rangle_{MD}$ are orthogonal. The requirements for such measurements are that the Hamiltonian (4) can be implemented and that the measuring device is initially in a state that is very narrow in Q compared to the interaction strength. Assuming all such Hamiltonians can be implemented we can make all ideal measurements with the following properties [8].

1. All systems initially in an eigenstate of A will remain unchanged by the measurement.

2. All systems not in an eigenstate of A will end up (after the measurement) in an eigenstate of $\hat{A}$ with and eigenvalue corresponding to the result of the measurement.

Now let us assume that such measurements could be made for non local observables such as $S_z^A S_z^B$ (the product of two spin 1 operators shared by two people, Alice and Bob) and show how we can send a superluminal signal from Bob to Alice thus endangering relativist causality. Before time $t = 0$ we prepare the two particles in the initial state

$$|\Psi\rangle_{in} = \frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)\,|0\rangle_B \tag{7}$$

We assume that at time $t = 0$ somebody performs an ideal measurement of $S_z^A S_z^B$. Immediately after $t = 0$ Alice performs a projective local measurement of the state $\frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)$. Bob, who has access to particle B, can send a superluminal

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signal to Alice in the following way. Just before $t = 0$ he decides to change the state of his spin to $|1\rangle_B$ or to leave it as it is $|0\rangle_B$. If he decides to do nothing, then non local measurement of $S_z^A S_z^B$ will not change the state of the particles because state (7) is an eigenstate of $S_z^A S_z^B$. Therefore, Alice, in her local projective measurement will find the state $\frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)$ with certainty. However, if Bob decides to change the state of his spin to $|1\rangle_B$ the initial state before the non local measurement at $t = 0$ will be changed to

$$|\Psi'\rangle_{in} = \frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)|1\rangle_B \tag{8}$$

It is not an eigenstate of $\sigma_{zA}\sigma_{zB}$ and thus after the measurement the system will end up in a mixed state of $|-1\rangle_A|1\rangle_B$ or $|0\rangle_A|1\rangle_B$. In both cases the probability to obtain a positive outcome in Alice's projective measurement on the state $\frac{1}{\sqrt{2}}(|-1\rangle_A + |0\rangle_A)$ is just half. Instantaneous change of probability of a measurement performed by Alice breaks causality, therefore instantaneous measurement of $S_z^A S_z^B$ is impossible.

So can we still measure non local variables? Let us first define non local variables as variables of a composite system with parts in two or more remote locations. For example the sum,product or modular sum of two spins. Such variables could always be measured in a non ideal measurement such as a demolition measurement [2]. This method does not contradict causality, since such methods do not project the system into the eigenstate corresponding to the measurement result.

For some non local variables such as that of a sum and a modular sum, ideal non local measurements do not contradict causality[4] and can be implemented

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using a non local (entangled) measuring device. The main problem (other then braking causality) of non local measurements is that no non local interactions exist in nature. If O_{AB} is a non local operator we cannot create the measurement Hamiltonian of the form (4) $H_M = g(t)PO_{AB}$ instead we can only use a Hamiltonian consisting of two local coupling terms.

$$H_m^{AB} = g_A(t)P_A A + g_B(t)P_B B \tag{9}$$

We assume $O_{AB} = O_{AB}(A, B)$ can be written as combination of the local operators A and B.(we are mainly interested in the sum $O_{AB} = A + B$ the product $O_{AB} = AB$ and the modular sum $O_{AB} = (A + B) mod \, N$ where N is some positive number). It is easy to see that such an interaction Hamiltonian requires two measuring devices (Q_A, Q_B), setting these measuring devices in the most naive way (i.e. two local Gaussian) will in turn give us two local measurements. For a sum and a modular sum we can set up the two measuring devices in the following ways in order to make the measurement non-local.

For the sum we have the measuring device initially in a state where $Q_A + Q_B = 0$ and $P_A - P_B = 0$ in this case the sum after the measurement will register on $Q_A + Q_B$. The modular sum is trickier but can be achieved if we place the measuring device in a initial state that has peaks at $(Q_A + Q_B) mod \, N = 0$ so that after the measurement it will be impossible to distinguish between the different peaks[2]. Another possible method is using a measuring device with a built in cyclic nature. An example is two spins in a maximally entangled symmetric state as described below.

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[2] A detailed description of such measurements is given in [4]

The most general state for two spin 1/2 particles can be described as

$$|\Psi\rangle_{AB} = \alpha|\uparrow\uparrow\rangle_{AB} + \beta|\downarrow\downarrow\rangle_{AB} + \gamma|\uparrow\downarrow\rangle_{AB} + \xi|\downarrow\uparrow\rangle_{AB} \tag{10}$$

we would generate a $(\sigma_Z^A + \sigma_Z^A) mod\ 4$ measurement by taking our measuring device in the state

$$|MD\rangle_{CD} = \frac{1}{\sqrt{2}}(|\uparrow\uparrow\rangle_{CD} + |\downarrow\downarrow\rangle_{CD}) \tag{11}$$

where C and A are in the same location and D and B are in the same location, so that the following unitary transformation is allowed

$$U = e^{i\frac{\pi}{4}(I^{AC} + \sigma_z^A \sigma_x^C + \sigma_x^C + I^{BD} + \sigma_z^B \sigma_x^D + \sigma_x^D)} \tag{12}$$

and this would be our measurement operation. It is basically a c-not gate on each of the pairs AC and BD so it flips the spin of the MD only if the state being measured is $\downarrow$. After the operation the final state would be.

$$\frac{1}{\sqrt{2}}\left(\alpha|\uparrow\uparrow\rangle_{AB} + \beta|\downarrow\downarrow\rangle_{AB}\right)\left(|\uparrow\uparrow\rangle_{CD} + |\downarrow\downarrow\rangle_{CD}\right) + \tag{13}$$

$$\frac{1}{\sqrt{2}}\left(\gamma|\uparrow\downarrow\rangle_{AB} + \xi|\uparrow\downarrow\rangle_{AB}\right)\left(|\uparrow\downarrow\rangle_{CD} + |\uparrow\downarrow\rangle_{CD}\right)$$

This is of course a good measurement of the modular sum. We must still look at the measuring device to reach the desired result, but this can be done locally at a later time (so Alice and bob can share the information).

The measurement procedures described above fulfill all the requirements of

ideal measurements and can be implemented on systems consisting of more then two particles with only minor adjustments. As mentioned above, given a large enough ensemble we can always use non-ideal measurements to measure the non local variables. These methods become more complicated when we move on to the pre and post selected systems as will be shown in section 5

3 The two state vector formalism (TSVF) of quantum mechanics.

The two-state vector formalism (TSVF) [8] is a time-symmetric description of the standard quantum mechanics originated in Aharonov, Bergmann and Lebowitz [9]. The TSVF describes a quantum system at a particular time by two quantum states: the usual one, evolving forward in time, defined by the results of a complete measurement at the earlier time, and by the quantum state evolving backward in time, defined by the results of a complete measurement at a later time.

According to the standard quantum formalism, an ideal (Von Neumann) measurement at time t of a non degenerate variable A tests for existence at this time of the forward evolving state $|A = a\rangle$ (it yields the outcome $A = a$ with certainty if this was the state) and creates the state evolving towards the future:

$$|\Psi(t')\rangle = e^{-\frac{i}{\hbar}\int_t^{t'} H dt}|A = a\rangle, \ \ t' > t. \tag{14}$$

In the TSVF this ideal measurement also tests for backward evolving state arriving from the future $\langle A = a|$ and creates the state evolving towards the past:

$$\langle\Phi(t'')| = \langle A = a|e^{\frac{i}{\hbar}\int_t^{t''} H dt}, \ \ t'' < t. \tag{15}$$

TSVF describes all quantum systems not only by the pre selected past or the post selected future, but by both. Using this formalism pure states can be described by their pre selected past state $|\Psi\rangle_i$(defined by a measurement to be made in the past), their post selected future state $\langle\Phi|_f$ (defined by a measurement made

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in the future) or by a two state vector $\langle\Phi|\ |\Psi\rangle$. For any strong measurement made at time t $(t_i < t < t_f)$ we can use the ABL formula

$$\text{Prob}(c_n) = \frac{|\langle\Phi|\mathbf{\Pi}_{C=c_n}|\Psi\rangle|^2}{\sum_j |\langle\Phi|\mathbf{\Pi}_{C=c_j}|\Psi\rangle|^2}. \tag{16}$$

to calculate the probability of eigenvalue c_n being the result of the measurement of the observable C. ($\Pi_{C=c_n}$ are projection operators projecting the state onto all eigenstates of C with the eigenvalue c_n). In many cases we can use this formula to predict the outcome of a measurement with certainty ($P(C = c_n) = 1$) even when both pre and post selected systems are not in an eigenstate of C (eg see sec 5) .[3]

Using the ABL formula, we can no longer use the bra-ket notation to express expectation values, instead we can speak of the expectation value of the measuring device. For pre and post selected systems the expectation value of the measuring device $\langle Q\rangle$, is given by

$$\langle Q\rangle = \sum_{all\ eignvalues} Prob(C = c_n) \times c_n \equiv \widetilde{\langle C\rangle} \tag{17}$$

Just by looking at this expression we can see that our usual relations between the expectation values of observables in pre selected systems don't hold for pre and post selected systems. As we will see later, this may cause a problem when trying to evaluate non-local observables using local strong measurements.

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[3] Another interesting example called the three box paradox is given in [8]

4 Weak measurements and weak values.

The most interesting phenomena which can be seen in the framework of the TSVF are related to weak measurements[14, 8]. Weak measurements are best described as Von Neumann measurements with a weakened coupling. After the interaction, all possible states of the measuring device relating to different eigenvalues overlap, i.e they are no longer orthogonal. Thus they cannot be distinguished from one another with certainty. At the limit where the states almost overlap completely, the total state of the MD and the system can be approximated as a product state. The system is left unchanged by the measurement and the measuring device points at some new value called a weak value [7]. It is not surprising that for pre selected only systems, this weak value is the expectation value, since it is just the mean result of all possible outcomes. For systems that are both pre and post selected however, the weak value is given by the complex number

$$(A_w) \equiv \frac{\langle \Phi | A | \Psi \rangle}{\langle \Phi | \Psi \rangle} \tag{18}$$

where as usual $|\Psi\rangle$ is the pre selected state, $\langle\Phi|$ is the post selected state and A is the observable measured. These weak values can be outside the allowed range of eigenvalues when the pre and post selected states are almost orthogonal $\langle\Phi|\Psi\rangle << 1$. This interesting phenomena is an effect of the pre and post selection causing some interference effects in the measuring device. These weak values have been observed experimentally [15, 16].

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4.1 The weak measurement process

For simplicity we describe the measuring device as a Gaussian around zero[4].

$$\psi_i^{MD}(Q) = (\Delta^2 \pi)^{-1/4} e^{-Q^2/4\Delta^2}. \tag{19}$$

The initial state before the measurement is given by $|\Psi\rangle_i |\psi\rangle_i^{MD}$

When we turn on the coupling Hamiltonian (4) the evolution of the state is given by the unitary operator $U = e^{-PA}$ (we neglect the rest of the Hamiltonian and choose $g(t)$ so that it gives 1 after integration). The intermediate state after the measurement is $U|\Psi\rangle_i |\psi\rangle_i^{MD}$. After post selection the whole system (MD and Ψ) is left in the state $_f\langle \Phi | U |\Psi\rangle_i |\psi\rangle_i^{MD} |\Phi\rangle_f$. For a weak enough measurement the measuring device will end up in the state

$$\psi_{fin}^{MD}(Q) \approx (\Delta^2 \pi)^{-1/4} e^{-(Q-A_w)^2/4\Delta^2} \tag{20}$$

so that it is now a Gaussian pointing at the weak value. We now move on to a more detailed description:

A system selected in an initial state

$$|\Psi\rangle = \sum_i \alpha_i |A = a_i\rangle \tag{21}$$

and coupled to a measuring device (19) via a coupling Hamiltonian (4) which gives rise to the unitary evolution $U = e^{-iPA}$, the system is later post selected in

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[4]Unless otherwise noted, the measuring device is always prepared with Δ being the uncertainty in Q.

the state

$$|\Phi\rangle = \sum_i \beta_i |A = a_i\rangle \tag{22}$$

so that we should be left with the (unnormalized) state

$$|\Phi\rangle \langle\Phi| \hat{U} |md\rangle |\Psi\rangle = (\Delta^2 \pi)^{-1/4} \left\{ \sum_i \alpha_i \beta_i^* e^{-\frac{(Q-a_i)^2}{4\Delta^2}} \right\} |\Phi\rangle \tag{23}$$

Since this is a product state we can write down the wave function of the measuring device as a pure state.

$$\psi_{fin}^{MD}(Q) = (\Delta^2 \pi)^{-1/4} \sum_i \alpha_i \beta_i + \left\{ \sum_i \sum_{m=1}^\infty \alpha_i \beta_i^* \left[\frac{-(Q-a_i)^2}{4\Delta^2} \right]^m / m! \right\} = \tag{24}$$

$$= (\Delta^2 \pi)^{-1/4} \langle\Phi|\Psi\rangle +$$

$$+ (\Delta^2 \pi)^{-1/4} \left\{ \sum_{m=1}^\infty \sum_i \alpha_i \beta_i^* \left[\frac{-(Q^2 + ka - 2Qka)}{4\Delta^2} \right]^m / m! \right\} =$$

$$(\Delta^2 \pi)^{-1/4} \left\{ \langle\Phi|\Psi\rangle + \sum_{m=1}^\infty \sum_i \alpha_i \beta_i^* \left[\frac{-(Q^2 + a_i^2 - 2Qa)}{4\Delta^2} \right] \right\} +$$

$$+ (\Delta^2 \pi)^{-1/4} \left\{ \sum_{m=2}^\infty \sum_i \alpha_i \beta_i^* \left[\frac{-(Q^2 + a_i^2 - 2Qa_i)}{4\Delta^2} \right]^m / m! \right\} =$$

$$(\Delta^2 \pi)^{-1/4} \langle\Phi|\Psi\rangle \left\{ 1 + \sum_{m=1}^\infty \left[\frac{-(Q - (A)_w)^2 + [(A^2)_w - (A)_w^2]}{4\Delta^2} \right] \right\} + \tag{25}$$

$$+ (\Delta^2 \pi)^{-1/4} \left\{ \sum_{m=2}^\infty \sum_i \alpha_i \beta_i \left[\frac{-(Q^2 + a_i - 2Qa_i)}{4\Delta^2} \right]^m / m! \right\} \approx$$

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$$(\Delta^2\pi)^{-1/4}\langle\Phi|\Psi\rangle\left[e^{-(Q-A_w)^2/4\Delta^2}+\frac{[(A^2)_w-(A)_w^2]}{4\Delta^2}\right]+O(\frac{1}{\Delta^4})\qquad(26)$$

For large Δ this is almost a Gaussian around the weak value. If the measurement was not disturbing, the normalization for this expression would be $\frac{1}{\langle\Phi|\Psi\rangle}$. The only non unitary operation is that of the post selection. But the measurement did slightly change the system, so this is only an approximation of the normalization. The term

$$\frac{[(A^2)_w-(A)_w^2]}{4\Delta^2}\qquad(27)$$

is the leading term which effects both the normalization and the position of the pointer and is thus the best quantitative measure of how weak the measurement really is. The nominator in this terms

$$[(A^2)_w-(A)_w^2]\qquad(28)$$

resembles the expression for the uncertainty squared. We call the square root of (28) the uncertainty in the weak value.

A more accurate value of the measuring device would be the state.

$$\psi_{fin}^{MD}(Q)\approx N\left\{(\Delta^2\pi)^{-1/4}\left[e^{-(Q-A_w)^2/4\Delta^2}+\frac{[(A^2)_w-(A)_w^2]}{4\Delta^2}\right]+O(\frac{1}{\Delta^4})\right\}$$
$$(29)$$

Where N is the normalization. As we will see later, these slight deviations from weak measurements are vital when comparing different types of measurements. We use them to define the "weakness" of the measurement.

Since weak values are complex, Q only points at the real part of the weak value, obtaining the complex part forces us to look at the conjugate momentum

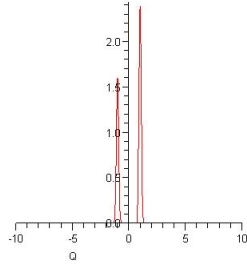P. Looking at the expectation values for P and Q we have.

$$Re(A_w) = \langle Q \rangle \tag{30}$$

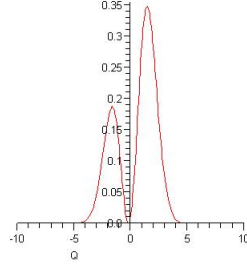$$Im(A_w) = 2\Delta^2 \langle P \rangle \tag{31}$$

Of course for a Gaussian, these expectation values are just the peak of the Gaussian which was shifted by the weak value from zero. Vaidman [17] argued that this obvious shift in the pointer is what allows us to consider weak values as "elements of reality".

The uncertainty in the measurement (Δ) could of course be reduced if we make a measurement of many such pre and post selected systems. For an ensemble of n such systems the uncertainty decreases as $\frac{1}{\sqrt{n}}$. This is of course the only practical way of making weak measurements since they inherently possess an uncertainty large enough to make one measurement unreliable . Now, since the accuracy of the measurement (the deviation from the weak value) depends on Δ and the overall uncertainty depend on Δ and n. We can ask ourselves a more practical question: How large an ensemble do we need to measure the weak value with a set deviation and a set uncertainty? We allow ourselves to be more precise and ask : How large an ensemble do we need to measure the weak value with a deviation of 1% and an uncertainty of 10% ? As will be seen in the next section, the size of this ensemble can be used to define an efficient and an inefficient measurement method.
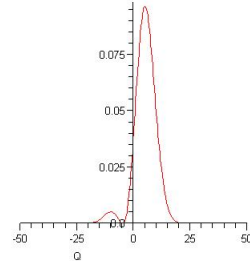
The above procedure has been derived for a measuring device described by a Gaussian, it has be shown [18]that any measuring device can be used for making such weak measurements so long as the interaction is weak enough. Thus we are
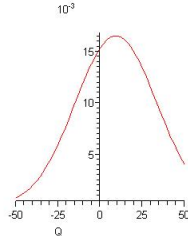
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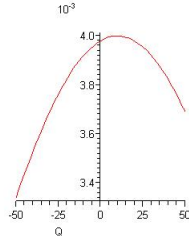(a) $\sigma = 0.1, \langle Q \rangle = 0.2$ (b) $\sigma = 1, \langle Q \rangle = 0.5$ (c) $\sigma = 5, \langle Q \rangle = 5$
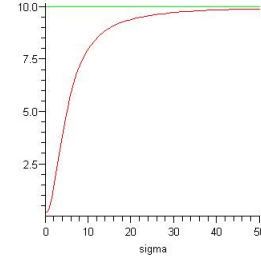
(d) $\sigma = 25, \langle Q \rangle = 9.6$ (e) $\sigma = 100, \langle Q \rangle = 9.98$ (f) $\langle Q \rangle vs \sigma$

Figure 1: A measurement of σ_x for the pre and post selected two state vector $[sin\epsilon \langle \uparrow_z \mid + cos\epsilon \langle \downarrow_z \mid]$ $\mid \uparrow_z \rangle$ with ϵ chosen so that the weak value $(\sigma_x)_w = 10$. The first 5 figures (a-e) are the probability density function for the pointer variable where sigma (the uncertainty) was varied from 0.1 in (a) corresponding to a strong measurement to 100 in (b) corresponding to a good weak measurement as can be seen from both the graph and the expectation value $\langle Q \rangle = 9.998$. Plot (f) is the expectation value for the measurement as a function of the uncertainty (σ)

reminded that a weak measurement of an observable A was defined as a standard Von Neumann measurement made weaker. The indirect (local) methods for non local measurements described bellow (in sections 6,7) require a more lenient definition of weak measurements. There are two requirements that must be fulfilled for the measurement to be a weak measurement.

1. The measuring device has to point at the weak value [17].

2. The measurement procedure must not be disturbing.

As we will see in sections 6,7, these requirements are again a little strong for indirect measurements since there is no measuring device pointing at the weak value. Still the first requirement has some interesting consequences when looking at ensembles of non identical systems.

4.2   weak measurements of a random ensemble of pre and post selected systems with the same weak value.

Usually when speaking of weak measurements, we talk of weak measurements made on an ensemble of identically prepared (pre and post selected) systems. An interesting change to this concept is weak measurements made on an ensemble of random systems. These systems should be pre and post selected in such a way that the weak value remains the same while the pre and post selections are changed. An example of this is an ensemble systems described by the two state vectors

$$\langle \Phi_j| \; |\Psi_j\rangle = \langle\uparrow|cos(\theta_j) + \langle\downarrow|sin(\theta_j) \quad cos(\phi_j)|\uparrow\rangle + sin(\phi_j)|\downarrow\rangle \qquad (32)$$

with $tan(\theta_j)tan(\phi_j) = \frac{1}{3}$ so that the weak value of σ_z is the same for all

systems.

$$(\sigma_z)_w = \frac{cos(\theta_j)cos(\phi_j) - sin(\theta_j)sin(\phi_j)}{cos(\theta)cos(\phi_j) + sin(\theta_j)sin(\phi_j)} = 0.5 \tag{33}$$

Since for weak measurements the final state is a Gaussian around the weak value, weak measurements made on all the systems will give us the same final state for the measuring device. The result of these weak measurements are indistinguishable from weak measurements made on and ensemble of identically prepared pre and post selected systems.

5 Non local measurements of pre and post selected systems.

If we are only interested in the result of the measurements rather then how much we do or do not disturb the original state it seems that one measurement scheme is as good as another. One could argue that such is the case for states that are pre and post selected since we already know that we succeeded in post selection. But since non ideal measurements project the state into a state other then the required eigenstate, the type of measurement does effect the outcome probabilities as can be seen by the ABL formula (16). So while joint local measurements made on non local systems would yield the correct outcomes for pre selected states, they don't do so for pre and post selected systems. A simple example is a system prepared in a state

$$|\Psi\rangle = \sqrt{\frac{1}{2+\epsilon^2}}(|\uparrow\rangle_A|\downarrow\rangle_B + |\downarrow\rangle_A|\uparrow\rangle_B) + \epsilon|\uparrow\rangle_A|\uparrow\rangle_B \qquad (34)$$

and post selected in the state

$$|\Phi\rangle = \sqrt{\frac{1}{2+\epsilon^2}}(|\uparrow\rangle_A|\downarrow\rangle_B - |\downarrow\rangle_A|\uparrow\rangle_B) + \epsilon|\uparrow\rangle_A|\uparrow\rangle_B \qquad (35)$$

with a measurement of $\sigma_z^A + \sigma_z^B$ made in the intermediate time.

For an ideal measurement of non local variable $\sigma_z^A + \sigma_z^B$ we have the following probabilities (using 16)

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$$Prob(\uparrow\uparrow)_{NL} = \frac{|\langle\Phi|\mathbf{\Pi}_{\uparrow\uparrow}|\Psi\rangle|^2}{|\langle\Phi|\mathbf{\Pi}_{\uparrow\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{\Pi}_{\downarrow\downarrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{\Pi}_{\sigma_z^A+\sigma_z^B=0}|\Psi\rangle|^2} = 1 \quad (36)$$

And

$$Prob(\downarrow\downarrow)_{NL} = Prob(\sigma_z^A + \sigma_z^B = 0)_{NL} = 0 \quad (37)$$

and using (17)

$$\langle\widetilde{\sigma_z^A + \sigma_z^B}\rangle = 2 \quad (38)$$

For ideal measurement of local variable σ_z^A, given that it is the only intermediate measurement that has been performed we have:

$$Prob(\uparrow)_A = \frac{|\langle\Phi|\Pi_{\sigma_z^A=\uparrow}|\Psi\rangle|^2}{|\langle\Phi|\Pi_{\sigma_z^A=\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\Pi_{\sigma_z^A=\downarrow}|\Psi\rangle|^2} = \frac{(\epsilon^2+1)^2}{2+\epsilon^4+2\epsilon^2} \quad (39)$$

$$Prob(\downarrow)_A = \frac{|\langle\Phi|\Pi_{\sigma_z^A=\downarrow}|\Psi\rangle|^2}{|\langle\Phi|\Pi_{\sigma_z^A=\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\Pi_{\sigma_z^A=\downarrow}|\Psi\rangle|^2} = \frac{1}{2+\epsilon^4-2\epsilon^2} \quad (40)$$

$$\widetilde{\langle\sigma_z^A\rangle} = \frac{2\epsilon^2+\epsilon^4}{2+\epsilon^4+2\epsilon^2} \quad (41)$$

The expectation value of σ_z^B is

$$Prob(\uparrow)_B = \frac{|\langle\Phi|\Pi_{\sigma_z^B=\uparrow}|\Psi\rangle|^2}{|\langle\Phi|\Pi_{\sigma_z^B=\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\Pi_{\sigma_z^B=\downarrow}|\Psi\rangle|^2} = \frac{(\epsilon^2-1)^2}{2+\epsilon^4-2\epsilon^2} \quad (42)$$

$$Prob(\downarrow)_B = \frac{|\langle\Phi|\Pi_{\sigma_z^B=\downarrow}|\Psi\rangle|^2}{|\langle\Phi|\Pi_{\sigma_z^B=\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\Pi_{\sigma_z^B=\downarrow}|\Psi\rangle|^2} = \frac{1}{2+\epsilon^4-2\epsilon^2}$$

$$\widetilde{\langle\sigma_z^B\rangle} = \frac{-2\epsilon^2+\epsilon^4}{2+\epsilon^4-2\epsilon^2} \quad (43)$$

It is easy to see that the expectation value of the non local variable (38) is very different from the sum of (41) and (43). However, it is more reasonable to compare (38) with the sum of expectation values of the outcomes of local measurements of σ_z^A and σ_z^B performed simultaneously (a joint measurement). This corresponds to measurement of a variable with non degenerate eigenstates $|\uparrow\rangle_A|\downarrow\rangle_B, |\downarrow\rangle_A|\uparrow\rangle_B, |\uparrow\rangle_A|\uparrow\rangle_B, |\downarrow\rangle_A|\downarrow\rangle_B$. In this case we have

$$Prob(\uparrow\uparrow)_{joint} = \frac{|\langle\Phi|\mathbf{\Pi}_{\uparrow\uparrow}|\Psi\rangle|^2}{|\langle\Phi|\mathbf{\Pi}_{\uparrow\uparrow}|\Psi\rangle|^2+|\langle\Phi|\mathbf{\Pi}_{\uparrow\downarrow}|\Psi\rangle|^2+|\langle\Phi|\Pi_{\downarrow\uparrow}|\Psi\rangle|^2} = \frac{\epsilon^2}{2+\epsilon^4} \tag{44}$$

$$Prob(\uparrow\downarrow)_{joint} = \frac{|\langle\Phi|\mathbf{\Pi}_{\uparrow\downarrow}|\Psi\rangle|^2}{|\langle\Phi|\mathbf{\Pi}_{\uparrow\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{\Pi}_{\uparrow\downarrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{\Pi}_{\downarrow\uparrow}|\Psi\rangle|^2} = \frac{1}{2+\epsilon^4}$$

$$Prob(\downarrow\uparrow)_{joint} = \frac{|\langle\Phi|\mathbf{\Pi}_{\downarrow\uparrow}|\Psi\rangle|^2}{|\langle\Phi|\mathbf{\Pi}_{\uparrow\uparrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{\Pi}_{\uparrow\downarrow}|\Psi\rangle|^2 + |\langle\Phi|\mathbf{\Pi}_{\downarrow\uparrow}|\Psi\rangle|^2} = \frac{1}{2+\epsilon^4}$$

$$\langle\widetilde{\{\sigma_z^A\} + \{\sigma_z^B\}}\rangle = \frac{\epsilon^2}{2+\epsilon^4} \tag{45}$$

The brackets $\{\}$ signify a separate measurement of each observable. We can easily see that not only do the measurement expectation values come out wrong, but when making local measurements we have a finite probability to get the results $\sigma_z^A + \sigma_z^B = 0$ corresponding to $\uparrow\downarrow$ and $\downarrow\uparrow$. These have zero probability for the non local measurement. If we had a large enough ensemble we could perform a complicated set of local measurements and find out the information about the pre and post selected states [19]. (Such measurements would often involve some intermediate measurement used to "delete" the past or the future.) So although the first statement made in this section may be true, there is a cost to making non

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ideal measurements.

Many measurement methods require an ensemble to improve accuracy. We can define an efficient measurement as one requiring a relatively small ensemble to get a good accuracy (as defined by us) and an inefficient measurement as one requiring a much larger ensemble for achieving the same accuracy. We can also say that an efficient measurement is one where size of the ensemble required to reach a certain accuracy depends on the uncertainty of the variable to be measured. An inefficient measurement would be one that depends on other parameters (such as the uncertainty of other variables). Going back to our example we see that the uncertainty is zero and therefore one measurement should be sufficient. Thus an efficient measurement method should require one measurement to arrive at the right result with a good accuracy. A few more measurements can be made to improve the statistical significance.

For weak measurements we can use the weak uncertainty (28) to classify our measurement scheme as efficient or inefficient. An efficient measurement would be one where the ensemble depends on the weak uncertainty as in the local weak measurements defined in section 4. As we will see in the next section local weak measurements of non local weak values depend on the local weak uncertainty rather then the non local ones, making them inefficient measurements.

6   Weak measurements of a non local sum.

We will start with the simplest case of a non local weak measurement, that of a sum of two observables. This case is relatively simple since there is a simple relation between the local and non local weak values.

$$(A + B)_w = A_w + B_w \tag{46}$$

It is just straightforward from (18). Using this formula, we have a simple method for measuring the non local weak value just by measuring the local weak values and adding the results. But as we will see this is an inefficient measurement (see sec 5) since the ensemble required to make a good precision measurement does not depend only on the uncertainty in $(A + B)_w$. A detailed derivation of this measurement process will give us some insight into the difference between the local measurements and the non local ones.

We start with the general state

$$|\Psi\rangle = \sum_i \alpha_{ij}|A = a_i, B = b_j\rangle \tag{47}$$

and coupled to a measuring device

$$\psi_{AB-in}^{MD}(Q) = (\Delta^2 \frac{\pi}{2})^{-1/2} e^{-(Q_A^2 + Q_B^2)/2\Delta^2} \tag{48}$$

via a coupling Hamiltonian which gives rise to the unitary evolution $\hat{U} = e^{-i(P_A A + P_B B)}$ (note that the overall uncertainty remains Δalthough each Gaussian

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is narrower by a factor of $\sqrt{2}$). The system is later post selected in the state

$$|\Phi\rangle = \sum_i \beta_{ij}|A = a_i, B = b_j\rangle \tag{49}$$

After post selection we are left with the unnormalized state

$$|\Phi\rangle \langle\Phi| \hat{U} |md\rangle |\Psi\rangle = (\frac{2}{\pi\Delta^2})^{1/2} \left\{ \sum_i \alpha_{ij}\beta_{ij} e^{-\frac{(Q_A-a_i)^2+(Q_B+b_j)^2}{2\Delta^2}} \right\} |\Phi\rangle \tag{50}$$

expanding the measuring device's wave function as a Taylor series we get

$$\psi_{fin}^{MD}(Q) \approx \left\{ (\Delta^2\pi)^{-1/4} \left[e^{-(Q-A_w)^2/4\Delta^2} + \langle\Phi|\Psi\rangle \frac{[(A^2)_w - (A)_w^2]}{2\Delta^2} + \langle\Phi|\Psi\rangle \frac{[(B^2)_w - (B)_w^2]}{2\Delta^2} \right] + C \right.$$
$$\tag{51}$$

it is not surprising that such a method depends on the local weak uncertainty rather then the non local one. (By local uncertainty we mean the uncertainty in local variables).

Using the above measuring scheme and the relation $(A + B)_w = A_w + B_w$ we can arrive at the weak value of the sum via local weak measurements alone. We will use the following example to show how this method differs from a direct (non local) measurement.

The state is prepared initially (pre selected) in the state

$$|\Psi\rangle_{AB} = \frac{1}{\sqrt{2\epsilon^2 + \delta^2 + 2}} |\uparrow\downarrow (1 + \epsilon) + \downarrow\uparrow (-1 + \epsilon) + \delta \uparrow\uparrow\rangle \tag{52}$$

and post selected in the state

$$|\Phi\rangle_{AB} = \frac{1}{2} |\uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow\rangle \tag{53}$$

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so that an intermediate weak measurement of $(\sigma_z^A + \sigma_z^B)$ would give

$(\sigma_z^A + \sigma_z^B)_w = \frac{\langle\uparrow\downarrow+\downarrow\uparrow+\uparrow\uparrow+\downarrow\downarrow|\sigma_z^A+\sigma_z^B|\uparrow\downarrow(1+\epsilon)+\downarrow\uparrow(-1+\epsilon)+\delta\uparrow\uparrow\rangle}{\langle\uparrow\downarrow+\downarrow\uparrow+\uparrow\uparrow|\uparrow\downarrow(1+\epsilon)+\downarrow\uparrow(-1+\epsilon)+\delta\uparrow\uparrow\rangle} = \frac{2\delta}{2\epsilon+\delta}$

Using this example we can now compare two measurement methods.

1. A joint local measurement (involving two local measuring devices) as described above.

2. A direct measurements (a non local measurement).

The density function of the measuring devices for the two measurement methods (joint local method and non local method) are plotted in figs 2 and 3 for different values of Δ(the uncertainty in measurement) with $\epsilon = -0.05$ and $\delta = 0.11$. This is an interesting example since the local uncertainty is much larger then the non local uncertainty. The plots are for each measuring device, i.e the non local measuring device in the case of the non local method and each of the local measuring devices in the case of the local method. For low values of Δ (fig 2) we can see that both methods don't approximate a Gaussian for the final state of the measuring device. Still it is obvious that the non local method converges much faster to a Gaussian then the local method, this is also true for larger values of Δ (fig 3).

Plots for the expectation values as a function of Δ are given in fig 4, again we see that the non local method converges a lot faster then the local method. If we want to make a precise measurements (deviation of 1% and uncertainty of 10%) we need an ensemble of $n = 2.2 \times 10^3$ systems for the non local method while a much larger ensemble $n = 8.2 \times 10^5$ is required for the joint local method. More examples with different δ and ϵ are given in fig 5 a,b,c. In fig:5 (d) we have a plot of the ensemble required to make a precise measurement of the same example with $\epsilon = -0.05$ and δ ranging from 0.1001 to 1.

We can see that for the general case of two measuring devices the measurement error is at best linearly proportional to the two local weak values. This type of measurement is an inefficient measurement regardless of whether we can derive the desired result after the measurement process.

For now , the non local measurement could have been seen as just an (unphysical) reference point since we did not yet describe a method for making such a non local measurement. We can still use this reference point to show that the measurements of non local values using local measuring devices are inefficient measurements. But we can build a good non local measuring device for measuring the non local weak sum. We will now describe two methods for setting up the entangled measuring device for this measurement. The first is a wide Gaussian around $Q_A + Q_B = 0$, $P_A - P_B = 0$ the second is an entangled measuring device made up of discrete overlapping Gaussian states so that again they obey $Q_A + Q_B = 0$ but each will be locally around different values. Here the local uncertainty of the measuring device is much larger then the non local uncertainty.

6.1 A Gaussian around $Q_A + Q_B = 0$, $P_A - P_B = 0$

We start by defining two orthogonal variables

$$Q^+ = Q_A + Q_B \tag{54}$$

$$Q^- = Q_A - Q_B \tag{55}$$

The initial state of the measuring device is

(a) $\Delta = 0.1$; $\langle Q \rangle = 1.1$

(b) $\Delta = 0.1$; $\langle Q_A \rangle = -0.10$

(c) $\Delta = 0.1$; $\langle Q_B \rangle = +0.10$

(d) $\Delta = 1$; $\langle Q \rangle = 1.2$

(e) $\Delta = 0.1$; $\langle Q_A \rangle = -0.06$

(f) $\Delta = 1$; $\langle Q_1 a \rangle = +0.07$

(g) $\Delta = 10$; $\langle Q \rangle = 11$

(h) $\Delta = 10$; $\langle Q_A \rangle = +0.5$
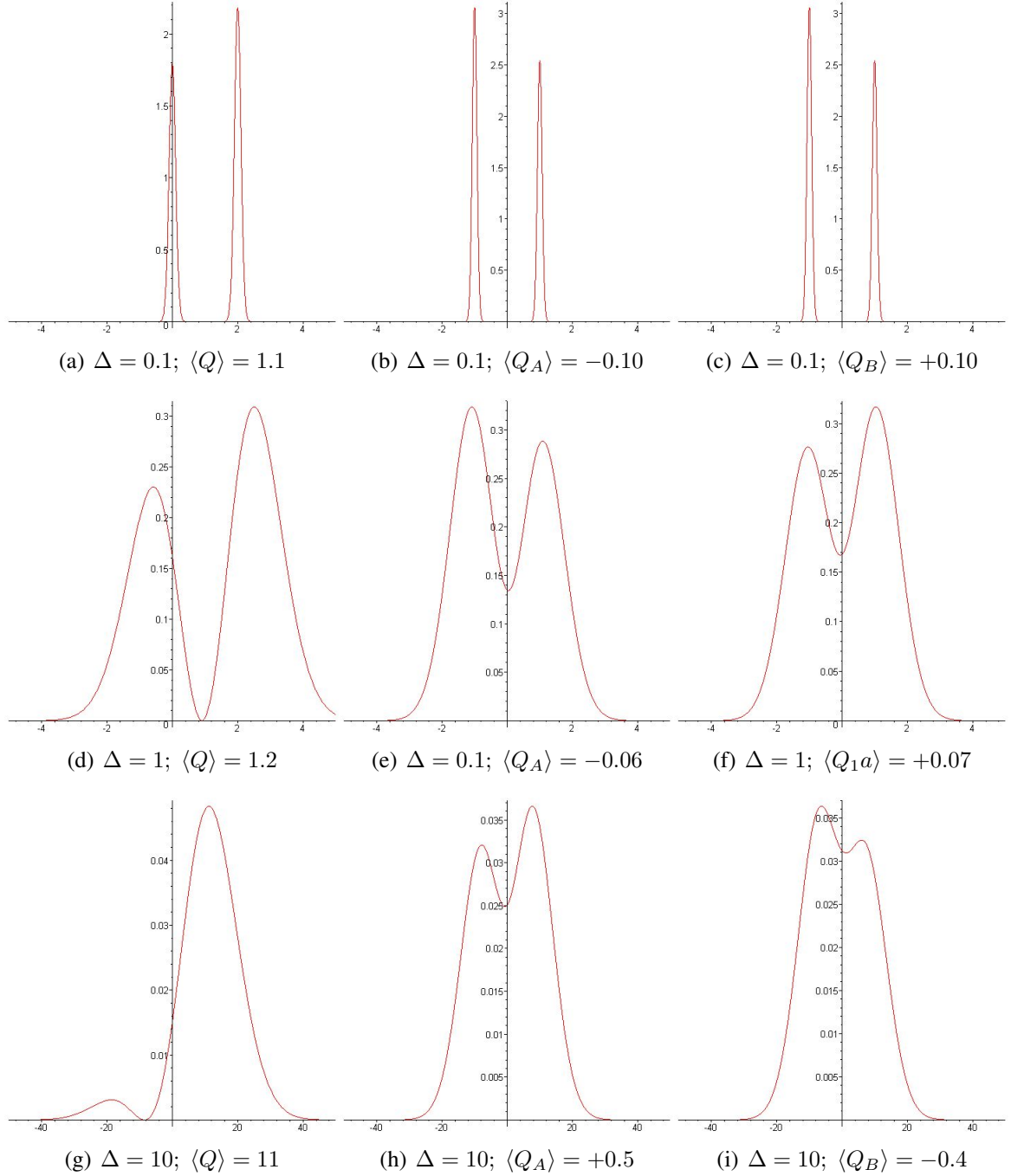
(i) $\Delta = 10$; $\langle Q_B \rangle = -0.4$

Figure 2: Comparing the local and non local methods for reaching the weak value of a sum. $(\sigma_z^A + \sigma_z^B)_w = \frac{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow | \sigma_z^A + \sigma_z^B | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle}{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle} = \frac{2\delta}{2\epsilon + \delta}$

For $\delta = 0.11$, $\epsilon = -0.05$ we have $(\sigma_z^A + \sigma_z^B)_w = 22$. The local weak values are $(\sigma_z^A)_w = 211$, $(\sigma_z^B)_w = -189$. The probability functions for each of the measuring devices are plotted for various values of Δ(the uncertainty in measurement) with the expectation value for the measurement given at the bottom. The left most column (a,d,g) is the non local measuring device, the center column (b,e,h) is the local measuring device at A, and the right column (c,f,i) is the local measuring device at B. In this figure we look at small values of Δ.

$\Delta = 0.1$ (a,b,c) is a strong measurement.

$\Delta = 1$ (e,f,g) is a slightly weaker measurement, where we can already see the interference effect of weak measurements.

For $\Delta = 10$ (h,i,j) we can see that the non local measuring device (j) has an expectation value outside the range of eigenvalues. (for $\Delta = 100, 100$ see fig 3)

(a) $\Delta = 100$; $\langle Q \rangle = 21.8$ (b) $\Delta = 100$; $\langle Q_A \rangle = 42$ (c) $\Delta = 100$; $\langle Q_B \rangle = -38$

(d) $\Delta = 1000$; $\langle Q \rangle = 21.998$ (e) $\Delta = 1000$; $\langle Q_A \rangle = 203$ (f) $\Delta = 1000$; $\langle Q_B \rangle = -181$
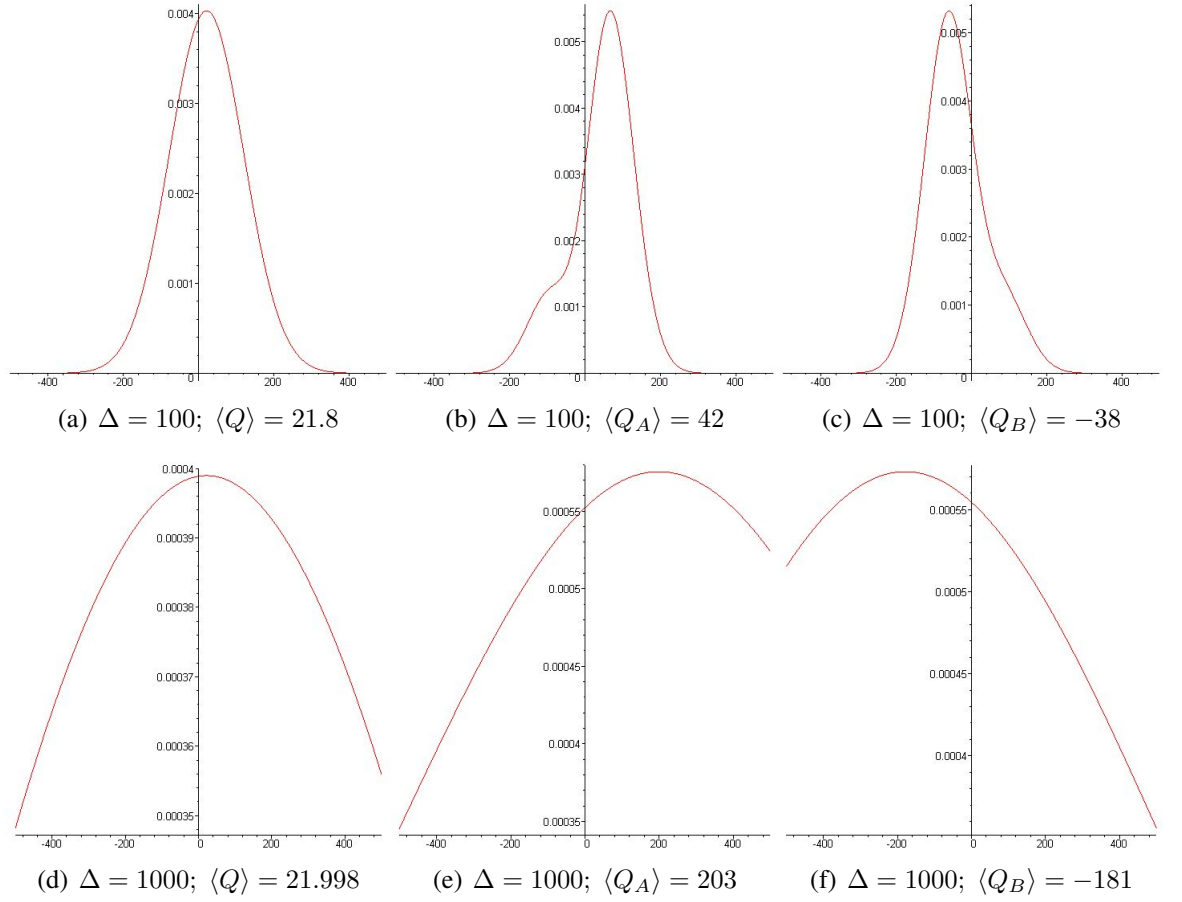
Figure 3: Comparing the local and non local methods for reaching the weak sum for $(\sigma_z^A + \sigma_z^B)_w = \frac{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow | \sigma_z^A + \sigma_z^B | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle}{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle} = \frac{2\delta}{2\epsilon + \delta}$

For $\delta = 0.11$, $\epsilon = -0.05$ we have $(\sigma_z^A + \sigma_z^B)_w = 22$ while the local weak values are $(\sigma_z^A)_w = 211$, $(\sigma_z^B)_w = -189$

The probability wave functions for each of the measuring devices are plotted for various values of Δ(the uncertainty in measurement) with the expectation value for the measurement given at the bottom. The left most column (a,d) is the non local measuring device, the center column (b,e) is the local measuring device at A, and the right column (c,f) is the local measuring device at B. In this figure we look at relatively large values of Δ. $\Delta = 100$ (a,b,c) is a good weak measurement for the non local measurement (a) while the non local values are still off and are still not a Gaussian. For $\Delta = 1000$ (e,f,g) we have Gaussian pointing close to the weak values for all three measurements. (for the stronger values$\Delta = 0.1, 1, 10$ see fig 2)
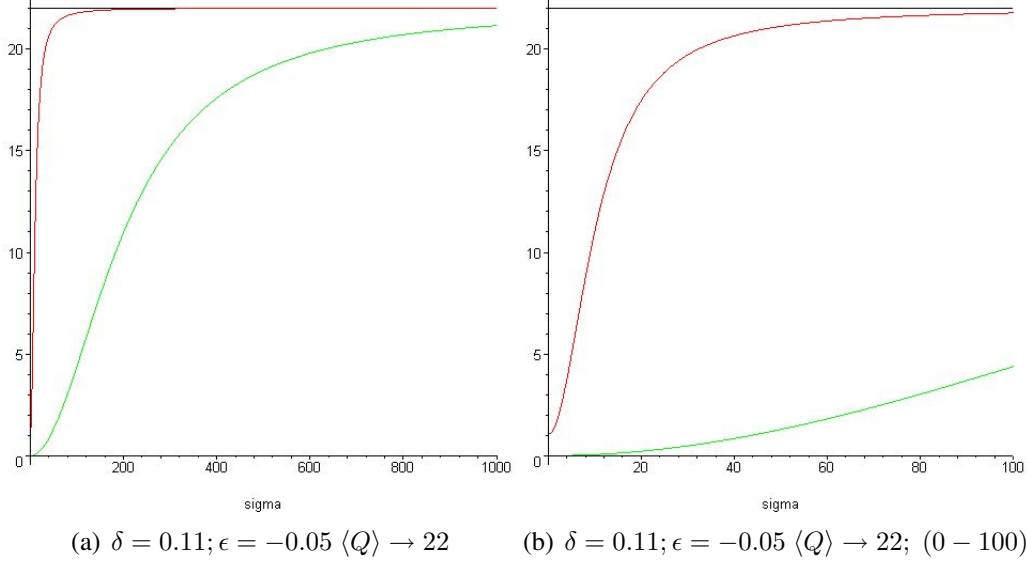
(a) $\delta = 0.11; \epsilon = -0.05 \langle Q \rangle \to 22$ (b) $\delta = 0.11; \epsilon = -0.05 \langle Q \rangle \to 22; \ (0 - 100)$

Figure 4: Comparing the local (green line) and non local (red line) methods for reaching the weak value of a sum. $(\sigma_z^A + \sigma_z^B)_w =$
$\frac{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow | \sigma_z^A + \sigma_z^B | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle}{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle} = \frac{2\delta}{2\epsilon + \delta}$
The plots are the expectation values at different ranges of the uncertainty (a, $\sigma =$ 0..1000 b, $\sigma = 0..100$). For $\delta = 0.11$, $\epsilon = -0.05$ we have $(\sigma_z^A + \sigma_z^B)_w = 22$. The local weak values are $(\sigma_z^A)_w = 211$, $(\sigma_z^B)_w = -189$ (plots for the probability density can be seen in fig 2,3)

It can easily be seen that the non local method (red line) converges much faster then the local method (green line). If we want to make a precise measurements (deviation of 1% and uncertainty of 10%) an ensemble of $n = 2.2 \times 10^3$ is required for the non local method while an ens amble of $n = 8.2 \times 10^5$ is required for the non local method.

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(a) $\delta = 0.1; \epsilon = 0 \ \langle Q \rangle \to 2$

(b) $\delta = -1; \epsilon = 0.6 \ \langle Q \rangle \to -10$

(c) $\delta = 1; \epsilon = 1 \ \langle Q \rangle \to 0.67$

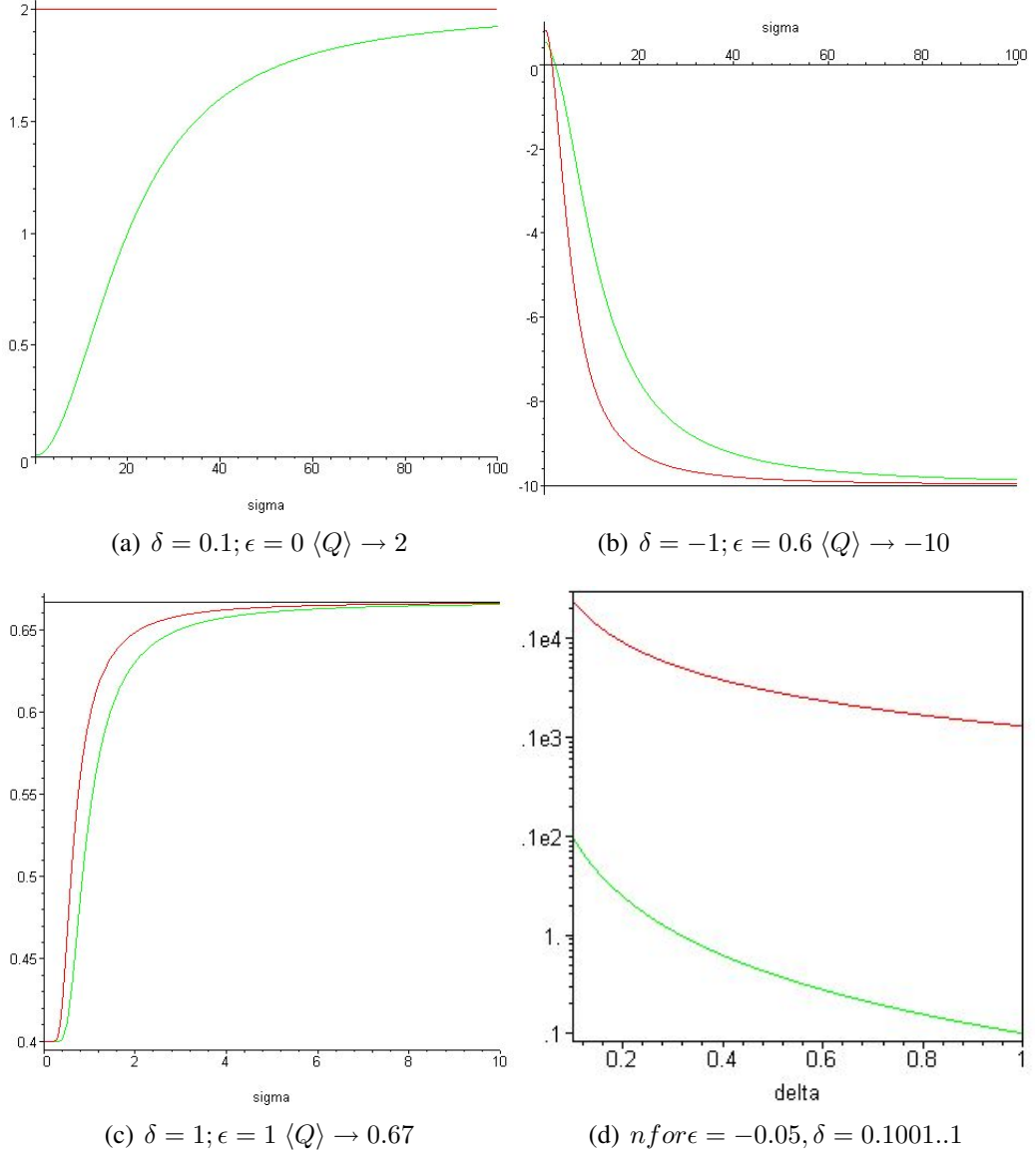(d) $n \, for \, \epsilon = -0.05, \delta = 0.1001..1$

Figure 5: Comparing the local (green line) and non local (red line) methods for reaching the weak value of a sum. $(\sigma_z^A + \sigma_z^B)_w =$ $\frac{\langle \uparrow\downarrow+\downarrow\uparrow+\uparrow\uparrow+\downarrow\downarrow|\sigma_z^A+\sigma_z^B|\uparrow\downarrow(1+\epsilon)+\downarrow\uparrow(-1+\epsilon)+\delta\uparrow\uparrow\rangle}{\langle \uparrow\downarrow+\downarrow\uparrow+\uparrow\uparrow|\uparrow\downarrow(1+\epsilon)+\downarrow\uparrow(-1+\epsilon)+\delta\uparrow\uparrow\rangle} = \frac{2\delta}{2\epsilon+\delta}$

The plots are the expectation values vs the uncertainty for different values of ϵ and δ corresponding to different local and non local weak values (non local=red line; local=green line).

a) $\delta = 0.1$, $\epsilon = 0$ - $(\sigma_z^A + \sigma_z^B)_w = 2$ while the local weak values are $(\sigma_z^A)_w = 21$, $(\sigma_z^B)_w = -19$. This is a special case where a strong non local measurement will with certainty give the value of 2 (as was seen in section 2).

b) $\delta = -1$, $\epsilon = 0.6$ - $(\sigma_z^A + \sigma_z^B) = -10$ while the local weak values are $(\sigma_z^A)_w = 5$, $(\sigma_z^B)_w = -15$. Even though the local and non local weak values are of the same order of magnitude, the non local method is clearly much better then the local one.

c) $\delta = 1$, $\epsilon = 1$ - $(\sigma_z^A + \sigma_z^B)_w = \frac{2}{3}$ while the local weak values are $(\sigma_z^A)_w = 1$, $(\sigma_z^B)_w = -\frac{1}{3}$. In this case there is a very small uncertainty in both the local and non local weak values. There is still some advantage to the non local measurement but it is very small.

d) A logarithmic plot of the size of the ensemble required for a precise measurement (deviation of 1% and uncertainty of 10%) for $\epsilon = -0.05$ and δ Going from 0.1001 to 1 (at $\delta = 1$ the uncertainty in the non local measurement is very low, we are almost certainly at the $\uparrow\uparrow$ state).

$$\psi_{AB-in}^{MD}(Q) = (\Delta\Sigma\pi)^{-1/2}e^{-(Q^+)^2/4\Delta^2}e^{-(Q^-)^2/4\Sigma^2} \tag{56}$$

We are interested in the limit $\Sigma \to \infty$. The uncertainty in Q^+ is Δ as always. Using this measuring device and the usual interactions $\hat{U}$ we have after pre selection

$$|\Psi\rangle = \sum_i \alpha_{ij}|A = a_i, B = b_j\rangle \tag{57}$$

measurement

$$\hat{U} = e^{-i(P_A A + P_B B)} \tag{58}$$

and post selection

$$|\Phi\rangle = \sum_i \beta_{ij}|A = a_i, B = b_j\rangle \tag{59}$$

the final state (where we just traced out Q^-)

$$|\Phi\rangle \langle\Phi| \hat{U} |md\rangle |\Psi\rangle = (\Delta^2\pi)^{-1/4} \left\{ \sum_i \alpha_{ij}\beta_{ij}e^{-\frac{(Q^+ - a_i - b_j)^2}{2\Delta^2}} \right\} |\Phi\rangle = \tag{60}$$

it is very simple to see that opening this as a Taylor series would give us a Gaussian around the non local weak value.

$$\psi_{fin}^{MD}(Q) \approx \left\{ (\Delta^2\pi)^{-1/4} \left[e^{-(Q^+ - A_w)^2/4\Delta^2} + \langle\Phi|\Psi\rangle \frac{[([A + B]^2)_w - (A + B)_w^2]}{2\Delta^2} \right] + O(\frac{1}{\Delta^4}) \right\}$$
$$\tag{61}$$

6.2 A sum of Gaussians.

Another method for making a non local measuring device is by setting it up in the following way

$$\psi_{in}^{MD}(Q) = \frac{1}{N} \sum_{l=0}^{k} e^{-\frac{(Q_1+l\xi)^2+(Q_2-l\xi)^2}{2\Delta^2}}$$

with

$$N^2 = \sum_{i=0}^{m} \sum_{j=0}^{m} \pi\Delta^2 exp[\frac{-\xi^2(i-j)^2}{2\Delta^2}]$$

this wave function is localized around $Q_A + Q_B = 0$ with the uncertainty being Δ while locally the uncertainty grows as k grows (depending on ξ). A good choice of the shift ξ seems to be $\xi = \Delta$ so that the Gaussian overlap increasing the uncertainty, but are removed from each other by enough so that the local uncertainty increases by a large amount. we end up with

$$\psi_{in}^{MD}(Q) = \frac{1}{\sqrt{\sum_{i=0}^{m} \sum_{j=0}^{m} \pi\Delta^2 exp[\frac{-(i-j)^2}{2}]}} \sum_{l=0}^{k} e^{-\frac{(Q_1+l\Delta)^2+(Q_2-l\Delta)^2}{2\Delta^2}}$$

The local uncertainty for k=0 is just $\frac{\Delta}{\sqrt{2}}$ as can be expected since this is just two local measuring devices. For other values of k the uncertainty grows as can be seen in fig 6.
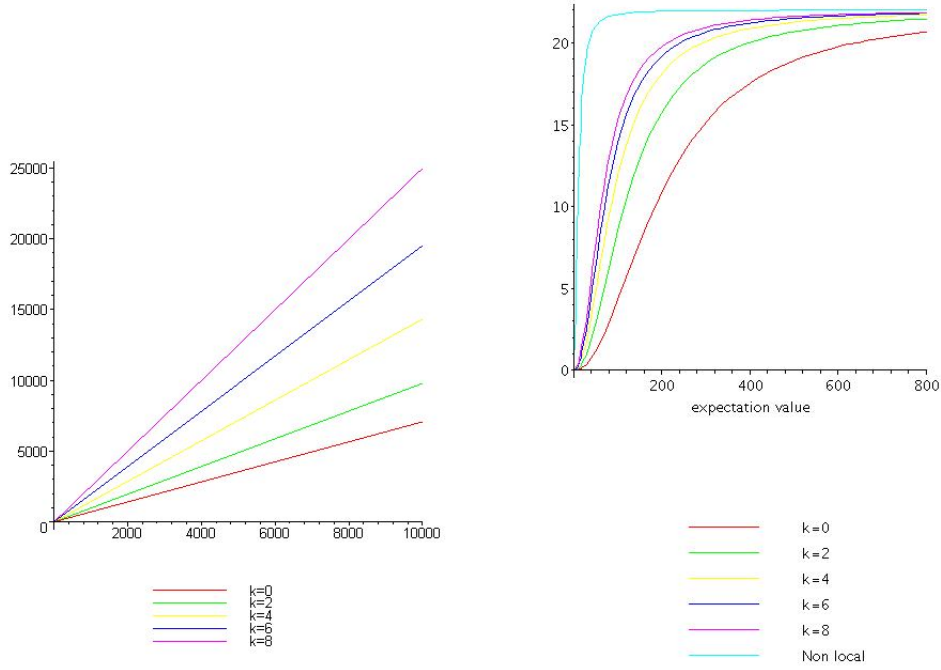
(a)

(b) $\delta = 0.11; \epsilon = -0.05 \, \langle Q \rangle \rightarrow 21;$

Figure 6: Comparing the local (red line), sum of Gaussian (green,yellow blue and purple lines) and non local (light blue line) methods for reaching the weak sum
$(\sigma_z^A + \sigma_z^B)_w = \frac{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow | \sigma_z^A + \sigma_z^B | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle}{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle} = \frac{2\delta}{2\epsilon+\delta}$
The plots are the expectation values vs the uncertainty with $\delta = 0.11$, $\epsilon = -0.05$ we have $(\sigma_z^A + \sigma_z^B)_w = 22$
in (a) we see the local uncertainty as a function of different values of k (for k=0 we have the standard local method). In (b) we can see that as k grows the measurement approaches the non local limit.

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7 Weak measurements of a non local product .

Recently Resch and Steinberg [10] devised a method for making measurements of the weak values of non local products. This method, uses local weak measurements to extract what they call "joint weak values" . The method was further developed by Lundeen and Resch [12] with the title "practical measurements joint weak values and their connection to the annihilation operator". It was later proposed by Mitchison, Jozsa and Popescu [13] that such measurements be used for sequential weak measurements . In this section we will investigate these methods and show that not only do they depend on the local uncertainty, but that the uncertainty in the final measurement is quadratic in Δ (the initial uncertainty of each Gaussian) rather then linear (as in the sum).

7.1 Joint weak values.

For a measuring device prepared in the state

$$\psi_{AB-in}^{MD}(Q) = (\Delta^2 \frac{\pi}{2})^{-1/2} e^{-(Q_A^2+Q_B^2)/4\Delta^2} \tag{62}$$

the general pre and post selection (57,59) and the measurement Hamiltonian $H_m = g(t)(P_A A + P_B B)$, we have the following formulas for obtaining the weak values (after the measurement).

$$Re(AB)_w = 2\langle Q_A Q_B \rangle - Re(A_w^* B_w) \tag{63}$$

$$Im(AB)_w = \frac{4\Delta^2}{\hbar}\langle Q_A P_B \rangle - Im(A_w^* B_w) \tag{64}$$

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$$Re(AB)_w = \langle Q_A Q_B \rangle - \frac{4\Delta^4}{\hbar^2} \langle P_A P_B \rangle \tag{65}$$

$$Im(AB)_w = \frac{2\Delta^2}{\hbar} \left(\langle Q_A P_B \rangle - \langle P_A Q_B \rangle \right) \tag{66}$$

These values depend on the correlations of the measuring devices and on the local weak values and are derived by using a second order expansion of the measuring device probability function $\psi^{MD}_{AB-fi}\psi^{MD*}_{AB-fi}$ with

$$\psi^{MD}_{AB-fi}(Q) \approx \{1 - A_w P_A - B_w P_B + \frac{1}{2}(A^2)_w P_A P_A + \frac{1}{2}(B^2)_w P_B P_B + \frac{1}{2}(AB)_w P_A P_B + O(\Delta^{-6})\}\psi^{MD}_{AB-in}(Q) \tag{67}$$

being the second order expansion of the final state of the measuring device ($P \sim (\Delta^{-2})$). The measuring device is prepared in such a way (62) that all inner products of an odd number of operators on the same particle ($\hat{Q}_A; \hat{P}_A$ or $\hat{Q}_A; \hat{P}_A$) go to zero so that we get the results (63-66).

Since for weak measurements the correlations for the two different measuring devices are very weak, there is only a very slight shift from zero in the different correlation functions. These slight shifts are of second order (in Δ^{-2}). But because the first order expectation value is zero, they are the leading terms. As can be expected the initial error for the measuring device is proportional to Δ^2 rather then Δ. For the initial state we have:

$$\Delta_{QQ} = \Delta^2 \tag{68}$$

$$\Delta_{PP} = \frac{1}{4\Delta^2} \tag{69}$$

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again the size of the ensemble (n) required for a precise measurement depends on the overall measurement error. This means that the ensemble for such joint weak measurements is much larger then the one required for the measurement of a sum. Using the example with the states presented previously (52,53)

$$(\sigma_z^A \sigma_z^B)_w = \frac{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow | \sigma_z^A \sigma_z^B | \uparrow\downarrow (1+\epsilon) + \downarrow\uparrow (-1+\epsilon) + \delta \uparrow\uparrow \rangle}{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow | \uparrow\downarrow (1+\epsilon) + \downarrow\uparrow (-1+\epsilon) + \delta \uparrow\uparrow \rangle} = \frac{\delta - 2\epsilon}{2\epsilon + \delta} \tag{70}$$

and again taking $\delta = 0.11$, $\epsilon = -0.05$ (fig 7) we can see for a precise measurements (deviation of 1% and uncertainty of 10%) an ensemble of $n = 1.5 \times 10^{12}$ is required for the joint weak measurements method. Since a method for making direct non local weak measurements has not yet been found, we compare this method with a non physical non-local measurement (involving non local interactions). Such a method would require an ensemble of $n = 2.5 \times 10^3$ for a precise measurement.

As we have shown, this joint weak measurement method is an inefficient measurement of the non local product. It is actually a lot worse then the joint measurement of a non local sum (compared with the direct non local weak measurement).

Although there is no method for directly measuring the non local product , the example above (70) is a special case where we can make a good measurement of the product by using a relation between the sum, the modular sum and the product. For a modular sum and a product (of two spin $1/2$ operators) we have the following relation:

$$\sigma_z^A \sigma_z^B = (\sigma_z^A + \sigma_z^B) \bmod 4 - 1 \tag{71}$$

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(a) $\delta = 0.11; \epsilon = -0.05 \langle Q \rangle \to 21$ (b) $\delta = 0.11; \epsilon = -0.05 \langle Q \rangle \to 21;$
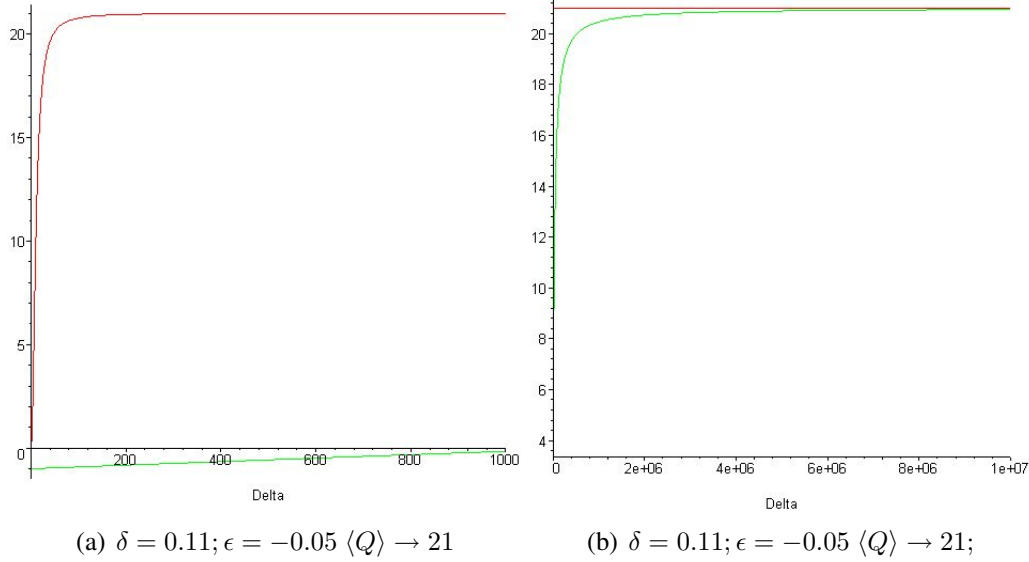
Figure 7: Comparing the local (green line) and non local [non physical] (red line) methods for reaching the weak value of a product. $\left(\sigma_z^A \sigma_z^B\right)_w = \frac{\langle\uparrow\downarrow+\downarrow\uparrow+\uparrow\uparrow+\downarrow\downarrow|\sigma_z^A\sigma_z^B|\uparrow\downarrow(1+\epsilon)+\downarrow\uparrow(-1+\epsilon)+\delta\uparrow\uparrow\rangle}{\langle\uparrow\downarrow+\downarrow\uparrow+\uparrow\uparrow|\uparrow\downarrow(1+\epsilon)+\downarrow\uparrow(-1+\epsilon)+\delta\uparrow\uparrow\rangle} = \frac{\delta-2\epsilon}{2\epsilon+\delta}$
The plots are the expectation values for different ranges of the uncertainty ($\Delta = 0..1000; \Delta = 0..10^7$). For $\delta = 0.11$, $\epsilon = -0.05$ we have $(\sigma_z^A \sigma_z^B)_w = 21$ while the local weak values are $(\sigma_z^A)_w = 211$, $(\sigma_z^B)_w = -189$ (plots for the probability density can be seen in fig 2,3)

It can easily be seen that the non local method (red line) converges much faster then the local method (green line). If we want to make a precise measurements (deviation of 1% and uncertainty of 10%) an ensemble of $n = 1.5 \times 10^{12}$ is required for the non local method while an ensemble of $n = 2.5 \times 10^3$ is required for the non local method.

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(a) $\delta = 0.1; \epsilon = 10 \langle Q \rangle \to 1$

(b) $\delta = 1; \epsilon = 0.55 \langle Q \rangle \to -101;$

(c) $\delta = -1; \epsilon = 0.6 \langle Q \rangle \to -11$

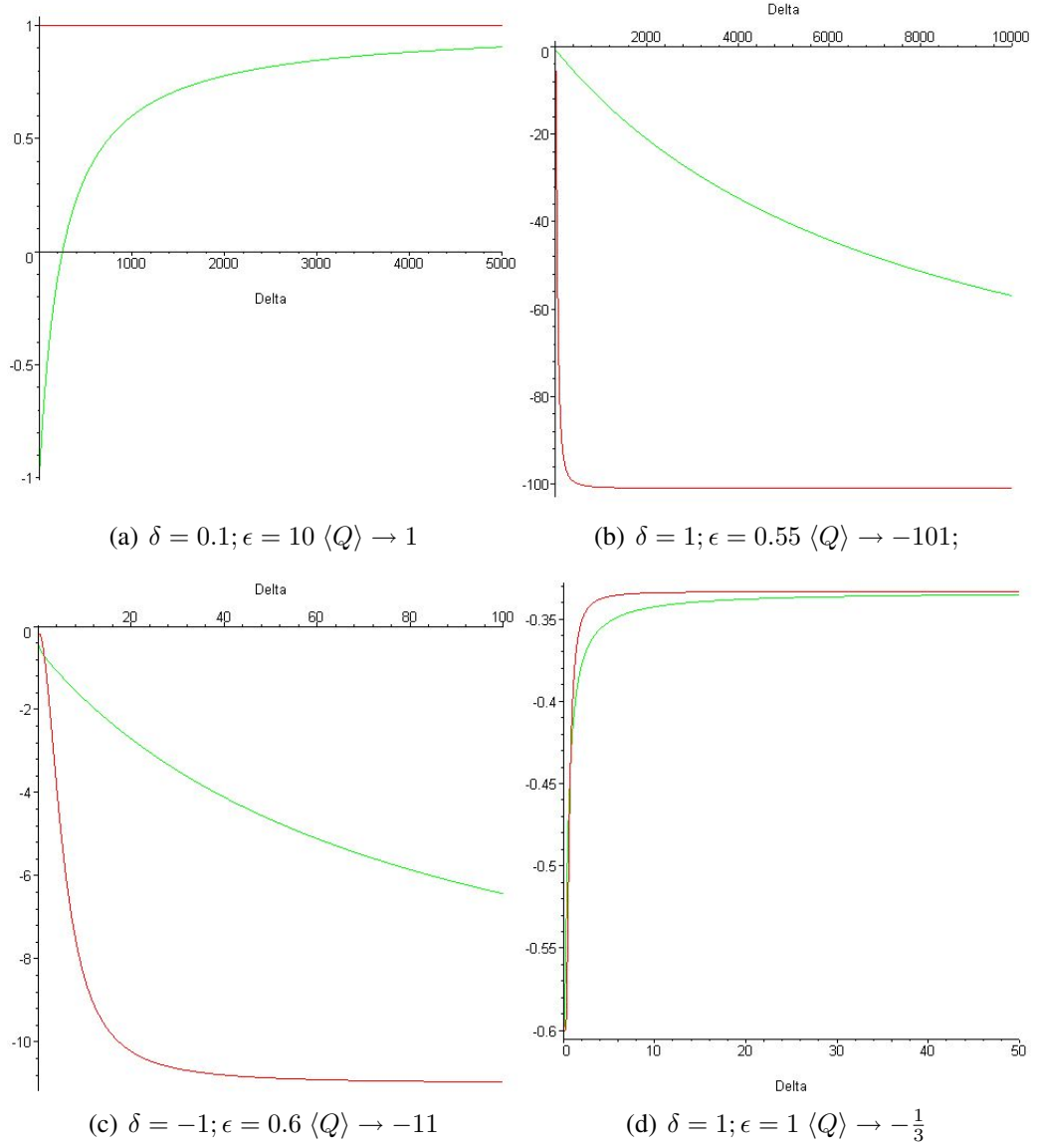(d) $\delta = 1; \epsilon = 1 \langle Q \rangle \to -\frac{1}{3}$

Figure 8: Comparing the local (green line) and non local [non physical] (red line) methods for reaching the weak value of a product. $(\sigma_z^A \sigma_z^B)_w = \frac{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow | \sigma_z^A \sigma_z^B | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle}{\langle \uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow | \uparrow\downarrow(1+\epsilon) + \downarrow\uparrow(-1+\epsilon) + \delta\uparrow\uparrow \rangle} = \frac{\delta - 2\epsilon}{2\epsilon + \delta}$
The plots are for different values of δ and ϵ.

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using this relation we can calculate the product from the modular sum. This is not much help since we don't know how to measure the modular sum directly, but since in our example there is zero probability for the ↓↓state, the sum is the same as the modular sum. Thus by measuring the sum we can actually get the value of the product. We already know how to measure the sum so we can use this method for making a good measurement of the weak value of the product. This method works for any two spin $1/2$ system, as long as one of the 4 possible local product states ↑↑; ↓↓; ↑↓;↓↑ has zero probability. (although we need to look at $\sigma_z^A - \sigma_z^B$ if its one of the spin zero states that has zero probability).

It is important to point out another practical consideration for the "joint weak measurements". As we have already shown, the pointer is set in such a way that it points at zero both before and after the interaction. The weak value is reached only by looking at second order effects. In a practical situation there might be a problem getting the measuring device set precisely on zero. Looking at (67) we can see that in the case where some of the first order terms are nonzero (after integration) we get contributions from A_w; B_w; $(A^2)_w$; $(B^2)_w$. For the standard weak measurements, Jozsa [18] showed that when using an arbitrary (weak) measuring device, the deviation from the weak value in the final result depends on properties of the measuring device and the weak value alone. This dependence on other weak values makes it even harder to make practical measurements of join weak values.

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7.2 Sequential weak measurements.

Weak values of a product were shown to be of interest by Mitchison, Jozsa and Popescu [13] when used for measuring the weak value of the product of the operators at different times, $(B_{t_2} A_{t_1})_w = \frac{\langle \Phi | B_{t_2} V A_{t_1} | \Psi \rangle}{\langle \Phi | \Psi \rangle}$ where V is the unitary evolution of the system between t_1 and t_2. The term used for the result is sequential weak values and it was shown that in a double interferometer experiment where the sequential strong measurement cannot be made because one measurement (at t_1) effects the other (at t_2), the sequential weak values can still be calculated. In that paper two methods were shown for calculating the sequential weak value of the number of photons going through an double interferometer. The first was the method of joint weak measurements which as we have shown cannot be considered a good measurement method. The second method involves three local weak measurements of different projection operators and some prior knowledge of the system. In that context it is interesting to discuss the meaning of the sequential weak value. Although the sequential weak value has a definite value, it cannot be measured directly and therefore cannot be considered an element of reality (as defined by Vaidman[17]) like standard weak measurements.

7.3 Weak measurements on ensembles of random non local systems with the same weak value.

The concept of weak measurements on random systems with the same weak value mentioned in sec 4.2 can also be used to examine joint weak values. Since both the values $\langle Q_A Q_B \rangle$ and $\langle P_A P_B \rangle$ depend on $Re(A_w^* B_w)$. A set of random states with the same value for $(AB)_w$ but different values for $(A_w^* B_w)$ will give us very

different values for $\langle Q_A Q_B \rangle$ and $\langle P_A P_B \rangle$. Since both cannot be measured at the same time, formula 65 cannot be used for calculating the weak value. Let us look at the following example: We start with an ensemble of systems pre selected in any of the states

$$|\Psi_i\rangle_{AB} = \frac{1}{\sqrt{2\epsilon_i^2 + \delta_i^2 + 2}} |\uparrow\downarrow (1 + \epsilon_i) + \downarrow\uparrow (-1 + \epsilon_i) + \delta_i \uparrow\uparrow\rangle \qquad (72)$$

and all post selected in the same state

$$|\Phi_i\rangle_{AB} = \frac{1}{2}|\uparrow\downarrow + \downarrow\uparrow + \uparrow\uparrow + \downarrow\downarrow\rangle \qquad (73)$$

With that the states $|\Psi_i\rangle$ chosen with $Re(\epsilon) = 0$, $Im(\epsilon) = \frac{-Im(\delta)}{2}$ and $Re(\delta) = 3Im(\delta)$ so that

$$(\sigma_z^A \sigma_z^B)_w = \frac{\delta - 2\epsilon}{2\epsilon + \delta} = 1 + \frac{2}{3}i \qquad (74)$$

defining $\delta'' \equiv Im(\delta)$

$$(\sigma_z^A)_w = \frac{\delta + 2}{2\epsilon + \delta} = 1 + \frac{2}{3}\delta'' + \frac{i}{3} \qquad (75)$$

$$(\sigma_z^B)_w = \frac{\delta + 2}{2\epsilon + \delta} = 1 - \frac{2}{3}\delta'' + \frac{i}{3} \qquad (76)$$

so that

$$(\sigma_z^A)_w^* (\sigma_z^B)_w = \frac{\delta + 2}{2\epsilon + \delta} = \frac{4}{3} + \frac{4\delta''^2}{9} + i\frac{4}{9}\delta''$$

Using (63-66) we have the following values for the joint measurements of the measuring devices.

$$\langle Q_A Q_B \rangle = \frac{7}{6} + \frac{2\delta''^2}{9} \qquad (77)$$

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$$\frac{4\Delta^4}{\hbar^2}\langle P_A P_B\rangle = \frac{1}{6} + \frac{2\delta''^2}{9} \tag{78}$$

$$\frac{2\Delta^2}{\hbar}\langle Q_A P_B\rangle = \frac{2}{6} + \frac{2}{9}\delta \tag{79}$$

$$\frac{2\Delta^2}{\hbar}\langle P_A Q_B\rangle = -\frac{2}{6} + \frac{2}{9}\delta \tag{80}$$

Since it is not possible to look at all these observables at the same time, their correlations will be lost if the spread δ'' is large enough.

If we allow ourselves to look at the measuring device in any (local) way, we can measure the value

$$\langle (Q_A - \frac{2\Delta^2}{\hbar}P_A)(Q_B + \frac{2\Delta^2}{\hbar}P_B)\rangle = Re(AB)_w + Im(AB)_w \tag{81}$$

which will give us the right result for $Im(AB)_w = 0$. In our case this will not be enough. As we already saw at the end of section 7.1, our example is one where we can measure the product by measuring the sum. Using this method we can make the measurement of the weak value for our random ensemble.

8   The meaning of non local weak values .

While eigenvalues, the outcomes of ideal strong measurements, and expectation values, the average of reading of ideal measurements over an ensemble of identically prepared systems are the basic textbook concepts of quantum mechanics, weak values have less solid foundations. We can still hear the echoes of the controversy of the days when weak values were introduced [20, 21, 22]. The justification of considering weak values as a description of a pre- and post-selected quantum system relies on the universality of influence of the coupling to a variable in the limit of its weakness. The pointer variable prepared in a natural way (see Jozsa for some limitations [18]) shifts due to weak measurement coupling as if it were coupled to a classical variable with the value equal to the weak value. (Note also that weak measurement performed on pre-selected ensemble show the expectation value even though we do not find the eigenvalues in the process of measurement.) Recently a method for making indirect measurements of weak variables was introduced [23]. This method together with the indirect methods mentioned in the last two sections remindes us that the weak value is the result of a calculation involving the pre selection, post selection and an hermitian operator.

We have shown that weak value have the following property: Although an ensemble is required to measure the weak value with any accuracy, this ensemble need not be made of identical systems. The only requirement is that the weak value remain the same for all systems. This is a property of weak measurements which is related to the effective coupling between the measuring device and the weak value. It shows that weak measurements are affected by the weak value alone and not by the other properties of the pre, and post selected states. Again

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we see that they are a property of the sytem that can be measured.

In this light we can see that those weak values which cannot be measured using a direct weak measurement have a lesser status. There is no method for creating an effective coupling between the measuring device and weak value of a product$(AB)_W$. Again we see that indirect measurements of non local weak values have a lesser status than standard weak measurements.

9 Conclusions

We discussed some methods for measuring the weak values of non local observables such as the sum of two observables (belonging to different particles) and the product of two observables (belonging to different particles). For the case of a non local sum we found a method for making non local weak measurements using an entangled measuring device. For the case of a product no such method has been found.

Indirect methods for reaching the non local weak value were analyzed and compared to direct methods. The size of the ensemble required to make a precise measurement was used to distinguish between efficient and inefficient methods. It was shown that for indirect methods the size of the ensemble required is not related directly to the uncertainty in the variable to be measured. These indirect methods are therefore inefficient.

When measuring a sum there are special cases where local measurements are almost as efficient as non local measurements, these are cases where the non local uncertainty is the same as the local uncertainty (28). For a small uncertainty in the non local variable and a large uncertainty in the local variable we showed (fig 2-4) that the non local method is much better then the local method.

The method of "joint weak values" for measureing the product is even more inefficient since even in the case where both the local and the non local uncertainty are small , local joint weak measurements require a large ensemble. This is because the deviation from zero in the observables $Q_A Q_B$, $P_A P_B$, $Q_A P_B$ and $P_A Q_B$ is very small compared with the width (the uncertainty) of these observables. For that reason such methods seem very impractical. However for some

specific cases, we found a way for measuring the weak value of the product using the method described for the weak measurement of a sum.

The interpretation of weak values as "elements of reality" [17] depends on the measuring device pointer being effectively coupled to the weak value. This effective coupling between the measuring device and the weak value allows us to measure the weak value using an ensamble of random systems with the same weak value. The above does not hold for joint weak measurements. It is therefore not clear if weak values of a non local product can be thought of as elements of reality.

Although we can write the expression for any weak value, for some non local weak values (such as a product) there is no efficient measurement procedure. It follows that those weak values which cannot be measured using direct weak measurements have a lesser status then those weak values that can be measured directly.

References

[1] L. Landau and R. Peierls. Erweiterung des unbestimmtheitsprinzips f?r die relativistische quantentheorie. *Z. Phys. 69, 56*.

[2] Lev Vaidman. Instantaneous measurement of nonlocal variables. *Phys. Rev. Lett.*, 90(1):010402, Jan 2003.

[3] Farid Ya Khalili Vladimir B. Braginsky. Quantum measurement, 1995.

[4] Yakir Aharonov, David Z. Albert, and Lev Vaidman. Measurement process in relativistic quantum theory. *Phys. Rev. D*, 34(6):1805–1813, Sep 1986.

[5] Yakir Aharonov and David Z. Albert. Can we make sense out of the measurement process in relativistic quantum mechanics? *Phys. Rev. D*, 24(2):359–370, Jul 1981.

[6] John von Neumann. *Mathematische Grundlagen der Quantenmechanik.* 1932. Reprinted: Dover Publications (1943); Presses Universitaires de France (1947); Madrid, Institute de Matematicas "Jorge Juan" (1949). Translated from German by Robert T. Beyer, Princeton University Press (1955).

[7] Yakir Aharonov, David Z. Albert, and Lev Vaidman. How the result of a measurement of a component of the spin of a spin-1/2 particle can turn out to be 100. *Phys. Rev. Lett.*, 60(14):1351–1354, Apr 1988.

[8] Yakir Aharonov and Lev Vaidman. The two-state vector formalism of qauntum mechanics, 2002.

[9] Yakir Aharonov, Peter G. Bergmann, and Joel L. Lebowitz. Time symmetry in the quantum process of measurement. *Phys. Rev.*, 134(6B):B1410–B1416, Jun 1964.

[10] K. J. Resch and A. M. Steinberg. Extracting joint weak values with local, single-particle measurements. *Phys. Rev. Lett.*, 92(13):130402, Mar 2004.

[11] K. J. Resch. Practical weak measurement of multiparticle observables. *J. Opt. B: Quantum Semiclass. Opt.*, 6:482–487, 2004.

[12] J. S. Lundeen and K. J. Resch. Practical measurement of joint weak values and their connection to the annihilation operator. *Physics Letters A*, 334:337–344, January 2005.

[13] G. Mitchison, R. Jozsa, and S. Popescu. Sequential weak measurement. *Pys Rev A*, 76(6):062105–+, December 2007.

[14] Yakir Aharonov and Lev Vaidman. Properties of a quantum system during the time interval between two measurements. *Phys. Rev. A*, 41(1):11–20, Jan 1990.

[15] G. J. Pryde, J. L. O'Brien, A. G. White, T. C. Ralph, and H. M. Wiseman. Measurement of Quantum Weak Values of Photon Polarization. *Physical Review Letters*, 94(22):220405–+, June 2005.

[16] N. W. M. Ritchie, J. G. Story, and R. G. Hulet. Realization of a measurement of a "weak value". *Physical Review Letters*, 66:1107–1110, March 1991.

[17] L. Vaidman. Weak-measurement elements of reality. *Foundations of Physics*, 26:895–906, July 1996.

[18] R. Jozsa. Complex weak values in quantum measurement. *Phys Rev A*, 76(4):044103–+, October 2007.

[19] L. Vaidman and I. Nevo. Nonlocal Measurements in the Time-Symmetric Quantum Mechanics. *International Journal of Modern Physics B*, 20:1528–1535, 2006.

[20] Y. Aharonov and L. Vaidman. Aharonov and vaidman reply. *Phys. Rev. Lett.*, 62(19):2327, May 1989.

[21] A. J. Leggett. Comment on ''how the result of a measurement of a component of the spin of a spin-(1/2 particle can turn out to be 100''. *Phys. Rev. Lett.*, 62(19):2325, May 1989.

[22] Asher Peres. Quantum measurements with postselection. *Phys. Rev. Lett.*, 62(19):2326, May 1989.

[23] L. M. Johansen. Reconstructing weak values without weak measurements. *Physics Letters A*, 366:374–376, July 2007.